Reconciling Wild Cards and Poker

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Appendix of Calculations

Counting Arguments Referred to in Paper

Counting Flush Hands using the Wild Card Rule, Jokers Wild There are 5,108 natural Flush hands. In order for the one wild card hand of the form $Wabcd$ to be declared a Flush, it must have Straight Flush and Flush as the two highest possible declarations. Thus, we need to count those hands with higher designation Straight Flush. The components to count are the number of ways to assign the lowest non-wild denomination (a), the number of ways to pick W, the number of ways to place b, c and d among the four remaining places, and the number of ways to assign a suit. Notice that the counting differs if W is used as the lowest card in the Straight Flush, and so it must be considered separately. There are $10 * \binom{2}{1} * \binom{4}{3} * \binom{4}{1} = 320$ hands whose lowest non-wild card is Ace through 10 and $1 * \binom{2}{1} * \binom{4}{4} * \binom{4}{1} = 8$ hands of the form $WJQKA$ for a total of 328 hands with one wild card declared a Flush under the Wild Card Rule. Hands of the form $WWabc$ whose higher designation is a Straight Flush are counted thus: $10 * \binom{2}{2} * \binom{4}{2} * \binom{4}{1} = 240$ hands with Ace through 10 as the lowest non-wild card; $1 * \binom{2}{2} * \binom{3}{2} * \binom{4}{1} = 12$ hands of the form $WJxxx$; and $1 * \binom{2}{2} * \binom{3}{3} * \binom{4}{1} = 4$ hands of the form $WWQKA$ for a total of 256. Thus, there are 5,692 total Flush hands under the Wild Card Rule.

Counting Three of a Kind Hands using the Wild Card Rule, Jokers Wild To count the Three of a Kind hands, first note there are 54,912 natural hands. Under the Wild Card Rule, hands of the form $Waabb$ are declared Three of a Kind, and there are 5,616 of them. With two wild cards, those hands of the form $WWabc$ whose higher designation is a Flush or a Straight are now declared Three of a Kind. There are $\binom{2}{2} * \binom{13}{3} * \binom{4}{1} - 256 = 888$ hands whose higher designation is a Flush (but not a Straight Flush). Counting those hands whose higher designation is a Straight is more involved. When the lowest non-wild card is Ace through 10, there are $10 * \binom{2}{2} * \binom{4}{2} * 4^3 = 3,840$ hands. When the lowest non-wild card is a Jack, there are $1 * \binom{2}{2} * \binom{3}{2} * 4^3 = 192$ hands. When the lowest non-wild card is a Queen, there are $1 * \binom{2}{2} * \binom{3}{3} * 4^3 = 64$ hands. Subtracting the 256 which are also Straight Flushes leaves 3,840 hands of this type. This gives a total of 65,256 Three of a Kind hands.

Counting Full House Hands under Wild Card Rule (Revised Rankings), Deuces Wild, form $WWWab$ Of the 3,568 hands whose higher rank is a Four of a Kind, we need to determine how many are declared Flush, Straight, or Full House hands under the Wild Card Rule. A hand becomes a Flush hand if a and b are the same suit but not close enough for a Straight Flush, so we have $\binom{4}{3} * \binom{12}{2} * \binom{4}{1} - 656 = 400$ hands of this type. This count chooses the three Ws, assigns denomination to a and b and picks a suit, subtracting the number of Straight Flush hands. To determine how many are declared a Straight, note that a and b must be close enough for a Straight, but not a Straight Flush. If the lowest non-wild card is an Ace, there are $\binom{4}{3} * \binom{3}{1} * 4^2$ hands (choosing W's, placing b, and assigning suits). If a is a 3 through 10, there are $8 * \binom{4}{3} * \binom{4}{1} * 4^2$ hands. If a is a Jack, we get $\binom{4}{3} * \binom{3}{1} * 4^2$ hands, and if a is a Queen there are $\binom{4}{3} * \binom{2}{1} * 4^2$ hands. Adding these and subtracting the 656 that are Straight Flush gives 1,968 hands declared Straights under the Wild Card Rule. Finally, $3568 - 400 - 1968 = 1200$ hands are declared to be Full House.

Counting Arguments Used to Complete Tables

In the arguments that follow, often the frequency is found using a product of combinations. For uniformity, the first combination will involve choosing the wild card W, the next combination counts ways to assign the denomination of non-wild cards denoted $a,\ b,\ c,\ d$, followed by combinations to assign the suits. When counting hands whose higher designation is a Straight, the first number is the number of ways to assign the lowest non-wild card.

Counting with Joker's Wild using the Wild Card Rule and Standard Hand Ranking Straight Flush: Only hands without wild cards end up here. There are 40 natural Straight Flush hands.

Four of a Kind: There are 624 natural hands. There are 26 hands of the form $Waaaa$ ($\binom{2}{1} * \binom{13}{1}$) and 52 hands of the form $WWaaa$ ($\binom{2}{2} * \binom{13}{1} * \binom{4}{3}$) for a total of 702 hands.

Flush: There are 5,108 natural Flush hands. A one wild card hand of the form $Wabcd$ must be a hand whose higher designation is Straight Flush. There are $10 * \binom{2}{1} * \binom{4}{3} * \binom{4}{1} = 320$ hands whose lowest non-wild card is Ace through 10 and $1 * \binom{2}{1} * \binom{4}{4} * \binom{4}{1} = 8$ hands of the form $WJQKA$ for a total of 328 hands with one wild card. Hands of the form $WWabc$ whose higher designation is a Straight Flush are counted thus: $10*\binom{2}{2}*\binom{4}{2}*\binom{4}{1} = 240$ hands with Ace through 10 as the lowest non-wild card; $1*\binom{2}{2}*\binom{3}{2}*\binom{4}{1} = 12$ hands of the form $WJxxx$; and $1 * \binom{2}{2} * \binom{3}{3} * \binom{4}{1} = 4$ hands of the form $WWQKA$ for a total of 256. Thus there are 5,692 total Flush hands.

Straight: There are 10,200 natural Straights. No wild card hands are declared Straights according to the Wild Card Rule.

Full House: There are 3744 natural hands. With wild cards, there are 4,992 hands of the form $Waaab$ and 3,744 hands of the form $WWaab$ for a total of 12,480 Full House hands.

Three of a Kind: There are 54,912 natural hands. Under the Wild Card Rule, hands of the form $Waabb$ are declared Three of a Kind, and there are 5,616 of them. With two wild cards, those hands of the form $WWabc$ whose higher designation is a Flush or a Straight are now declared Three of a Kind. There are $\binom{2}{2} * \binom{13}{3} * \binom{4}{1} - 256 = 888$ hands whose higher designation is a Flush (but not a Straight Flush). Counting those whose higher designation is a Straight is more involved. When the lowest non-wild card is Ace through 10, there are $10 * \binom{2}{2} * \binom{4}{2} * 4^3 = 3,840$ hands. When the lowest non-wild card is a Jack, there are $1 * \binom{2}{2} * \binom{3}{2} * 4^3 = 192$ hands. When the lowest non-wild card is a Queen, there are $1 * \binom{2}{2} * \binom{3}{3} * 4^3 = 64$ hands. Subtracting the 256 which are also Straight Flushes leaves 3,840 hands of the type $WWabc$ designated Three of a Kind under the rule. This gives a total of 65,256 Three of a Kind hands.

Two Pair: There are 123,552 natural Two Pair hands. With one wild card, there are 164,736 hands of the form $Waabc$. With two wild cards, we add 13,320 hands of the form $WWabc$ (found by $\binom{2}{2} * \binom{13}{3} * 4^3 = 18304$ total such hands, less (3840+888+256) such hands already designated as higher hands). These total 301,608 Two Pair hands.

One Pair: We have 1,098,240 natural hands. Hands of the form $Wabcd$ whose higher designation is a Flush or a Straight are also counted as One Pair. There are $\binom{2}{1} * \binom{13}{4} * \binom{4}{1} - 328 = 5392$ hands whose higher designation is a Flush (less those with higher designation Straight Flush). There are $10 * \binom{2}{1} * \binom{4}{3} * 4^4 + 1 * \binom{2}{1} * \binom{4}{4} * 4^4 - 328 = 20,664$ hands whose higher designation is a Straight, less those with higher designation Straight Flush. There are no hands with two wild cards designated as One Pair, thus the total is 1,124,296.

Counting with Joker's Wild using the Wild Card Rule and Revised Hand Ranking There are no changes to the counting arguments in this case!

Counting with Deuces Wild using the Wild Card Rule and Standard Hand Ranking Note that the method for counting Straights changes. AW345 is considered a Straight and must be counted separately each time.

Five of a Kind: No hands.

Straight Flush: There are 32 natural hands. Hands of the form $WWWWa$, of which there are $48 = \binom{4}{4} * \binom{48}{1}$, are also designated Straight Flush hands, for a total of 80.

Four of a Kind: There are 528 natural hands. There are 48 hands of the form $Waaaa$ ($\binom{4}{1} * \binom{12}{1} * \binom{4}{4}$). There are 288 hands of the form $WWaaa$ ($\binom{4}{2} * \binom{12}{1} * \binom{4}{3}$). With three wild cards, hands may be either of the form $WWWaa$ with higher designation Five of a Kind or $WWWab$ with higher designation Straight Flush. There are 288 hands of the form $WWWaa$ ($\binom{4}{3} * \binom{12}{1} * \binom{4}{2}$) and 656 hands $WWWab$. As usual, the counting for $WWWab$ has more steps. When the lowest non-wild card is and Ace, there are $1 * \binom{4}{3} * \binom{3}{1} * \binom{4}{1} = 48$ hands. When the lowest non-wild card is 3 through 10, we have $8 * \binom{4}{3} * \binom{4}{1} * \binom{4}{1} = 512$ hands. If the lowest non-wild card is a Jack, there are $1 * \binom{4}{3} * \binom{3}{1} * \binom{4}{1} = 48$ hands. If the lowest non-wild card is a Queen, there are $1 * \binom{4}{3} * \binom{2}{1} * \binom{4}{1} = 32$ hands and if the lowest non-wild card is a King, we have $1 * \binom{4}{3} * \binom{1}{1} * \binom{4}{1} = 16$ hands. This gives a total of 656 hands of the form $WWWab$ designated as Four of a Kind under the Wild Card Rule. The total number of Four of a Kind hands is then 1,808.

Full House: There are 3168 natural Full House hands. Hands of the form $Waaab$, of which there are $\binom{4}{1} * \binom{12}{1} * \binom{4}{3} * 44 = 8,448$ are also designated Full House. With two wild cards, hands of the form $WWaab$ give $\binom{4}{2} * \binom{12}{1} * \binom{4}{2} * 44 = 19,008$ additional hands. With three wild cards, we consider hands of the form $WWWab$ whose higher designation is Four of a Kind. There are $\binom{4}{3} * \binom{12}{2} * 4^2 - 656 = 3568$ hands of this type (subtracting those that would have Straight Flush as the higher designation). This gives a total of 34,192 Full House hands.

Flush: There are 3,136 natural hands. Hands of the form $Wabcd$ that have a higher designation of Straight Flush will be ruled Flush hands. There are 16 hands with an Ace as the lowest non-wild card ($\binom{4}{1} * \binom{4}{4} * \binom{4}{1}$) and $8 * \binom{4}{1} * \binom{4}{3} * \binom{4}{1} = 512$ hands with 3 through 10 as the lowest non-wild card. Next we have $1 * \binom{4}{1} * \binom{3}{3} * \binom{4}{1} = 16$ hands with a Jack as the lowest non-wild card. This gives 544 hands of the form $Wabcd$. With two wild cards, we need to count hands of the form $WWabc$ that have a higher designation of Straight Flush. Here we have $\binom{4}{2} * \binom{3}{2} * \binom{4}{1} = 72$ hands with Ace low. Next, there are $8 * \binom{4}{2} * \binom{4}{2} * \binom{4}{1} = 1152$ hands having 3 through 10 low. Then we have $1 * \binom{4}{2} * \binom{3}{2} * \binom{4}{1} = 72$ hands with Jack low, followed by $1 * \binom{4}{2} * \binom{2}{2} * \binom{4}{1} = 24$ hands with Queen low. This yields 1320 hands with two wild cards. Together, we get 5,000 Flush hands. (Note that hands with three or four wild cards will never be designated a Flush under the Wild Card Rule.)

Straight: There are 8,160 natural Straight hands. No wild card hands are designated as Straights under the Wild Card Rule with this ranking.

Three of a Kind: There are 42,240 natural hands. Hands of the form $Waabb$, of which there are $\binom{4}{1} * \binom{12}{2} * \binom{4}{2} * \binom{4}{2} = 9504$ are also Three of a Kind. With two wild cards, hands of the form $WWabc$ which have a Straight or Flush as a higher designation also must be counted. Let's first address the Straights. There are $\binom{4}{2} * \binom{3}{2} * 4^3$ having Ace low. With 3 to 10 low, we get $8 * \binom{4}{2} * \binom{4}{2} * 4^3$ more. When Jack is low, we add $1 * \binom{4}{2} * \binom{3}{2} * 4^3 = 1152$ and with Queen low we have $1 * \binom{4}{2} * 1 * 4^3 = 384$ more. Then, we must subtract the 1320 Straight Flushes, so we have 19,800 hands with higher designation Straight. There are $\binom{4}{2} * \binom{12}{3} * \binom{4}{1} - 1320 = 3960$ hands with higher designation Flush. Thus, there are 75,504 Three of a Kind hands.

Two Pair: There are 95,040 natural hands. With one wild card, there are $\binom{4}{1} * \binom{12}{1} * \binom{4}{2} * \binom{11}{2} * 4^2 = 253,440$ hands of the form $Waabc$ designated Two Pair. Hands of the form $WWabc$ that have higher designation Three of a Kind must also be counted here. There are $\binom{4}{2} * \binom{12}{3} * 4^3 = 84,480$ total such hands, but we must discount those of higher ranking. Thus, we subtract 19,800 Straights, 3960 Flushes, 1320 Straight Flushes for a total of 59,400. Adding these gives 407,880 Two Pair hands.

One Pair: There are 760,320 natural One Pair hands. The only wild-card hands counted here are of the form $Wabcd$ whose higher designation is Flush or Straight. For Flush, we have $\binom{4}{1} * \binom{12}{4} * \binom{4}{1} - 544$ after subtracting the Straight Flush hands. In the Straight category, there are $\binom{4}{1} * \binom{4}{4} * 4^4$ Ace low hands, $8 * \binom{4}{1} * \binom{4}{3} * 4^4$ hands with 3 through 10 low and $\binom{4}{1} * 4^4$ Jack low hands. This total, less the 544 Straight Flush hands gives 34,272. These give a total of 801,968 One Pair hands.

Counting with Deuces Wild using the Wild Card Rule and Revised Hand Ranking With this ranking, the only type of hand that changes designation is of the form $WWWab$ whose higher designation is Four of a Kind. Originally, all hands of this type (3,568 of them) were ruled to be Full House hands. With the revised ranking of hands, some will now be Flush hands, some Straights, and the remaining hands become Full House hands.

If $WWWab$ have a and b in the same suit but not close enough in denomination to be a Straight Flush, the Wild Card Rule designates them Flush. There are $\binom{4}{3} * \binom{12}{2} * \binom{4}{1} - 656 = 400$ such hands.

If a and b are different suits and close enough in denomination to be a Straight, the Wild Card Rule designates them Straight. There are $1 * \binom{4}{3} * \binom{3}{1} * 4^2 + 8 * \binom{4}{3} * \binom{4}{1} * 4^2 + \binom{4}{3} * \binom{3}{1} * 4^2 + \binom{4}{3} * \binom{2}{1} * 4^2 - 656 = 1,968$ such hands. Again here, we count the cases Ace low, 3 to 10 low, Jack low and Queen low separately and then subtract those that would be Straight Flush hands.

The remaining $3568 - 400 - 1968 = 1200$ hands become Full House hands.

The rest of the counting remains unchanged in the revised ranking, hence the Table given.

Counting with Two Jacks Wild using the Wild Card Rule and the Standard Hand Ranking Five of a Kind: 0

Straight Flush: There are 32 natural hands of the form $abcde$. To count these, note that there are two cases: when a is Ace through 6 and when a is 7 through 10 (necessitating a non-wild Jack). There are $6 * 1 * 4$ of the first and $4 * 1 * 2$ of the second (only two suits for the Jack and thus for the natural Straight Flush).

Four of a Kind: There are 552 natural hands $(\binom{12}{1} * \binom{4}{4} * 46)$, 24 of the form $Waaaa$ and 48 of the form $WWaaa$, for a total of 624.

Full House: There are 3,216 natural hands of the form $aaabb$. To count these, note we must consider two cases. When b is not a Jack (and neither is a) we have $12P2 * \binom{4}{3} * \binom{4}{2} = 3168$ hands (note the use of a permutation here, as it matters which is the triple and which is the pair). When b is a Jack, we have $\binom{2}{2} * \binom{12}{1} * \binom{4}{3} = 48$ hands. With one wild card, $Waaab$ is the form of the hand. There are $\binom{2}{1} * \binom{12}{1} * \binom{4}{3} * 46 = 4416$ hands of this form. Hands with two wild cards look like $WWaab$ and there are again two cases to consider. If a is not a Jack, then there are $\binom{2}{2} * \binom{12}{1} * \binom{4}{2} * 46$ hands. If a is a Jack, there are $\binom{2}{2} * \binom{2}{2} * 48$ hands, for a total of 3,360 hands of the form $WWaab$. This gives a total of $10,992$ Full House hands.

Flush: There are 4,126 natural Flush hands. These hands have the form $abcde$. To count them, note that there are two cases: none of a, b, c, d or e is a Jack, or one is a Jack. There are $\binom{12}{5} * 4$ of the first case and $\binom{2}{1} * \binom{12}{4}$ of the second (note that the choice of Jack determines the suit for the rest). After subtracting the 32 Straight Flush hands, we have $3168 + 990 - 32 = 4,126$ natural hands. With one wild card, the hand $Wabcd$ must have a higher ranking of Straight Flush. To count this, we must consider four separate cases: a is Ace to 6, a is 7 to 10 and b is a Jack, a is 7 to 10 and there is no non-wild Jack, and a is a non-wild Jack. In that order, there are $6 * \binom{2}{1} * \binom{4}{3} * \binom{4}{1} + 4 * \binom{2}{1} * \binom{2}{1} * \binom{3}{2} + 4 * \binom{2}{1} * 1 * 4 + 1 * \binom{2}{1} * 1 * 2 = 192 + 48 + 32 + 4 = 276$ hands of this type. For two wild cards, hands of the form $WWabc$ must have a higher ranking of Straight Flush. We have the same four cases as above, with the additional case of a being a Queen. There are $6 * \binom{2}{2} * \binom{4}{2} * 4 + 4 * \binom{2}{2} * \binom{2}{1} * \binom{3}{1} + 4 * \binom{2}{2} * \binom{3}{2} * 4 + 1 * \binom{2}{2} * \binom{2}{1} * \binom{3}{2} + 1 * \binom{2}{2} * 1 * 4 = 144 + 24 + 48 + 6 + 4 = 226$ hands of this form. Thus, there are a total of 4,628 Flush hands.

Straight: Only natural Straights count here. There are two cases to consider: when a is Ace through 6 and when a is 7 through 10. There are $6 * 1 * 4^5$ of the first type and $4 * 1 * 4^4 * 2$ of the second type (there are only two Jack suits). Less the Straight Flush hands leaves $6144 + 2048 - 32 = 8160$ Straight hands.

Three of a Kind: Natural Three of a Kind hands, $aaabc$ cannot have a as Jack, but could have one of b or c as a Jack. Thus there are $\binom{12}{1} * \binom{4}{3} * [\binom{11}{2} * 4^2 + 2 * 44] = 46,464$ natural hands. With one wild card, the hand has the form $Waabb$. Two cases are when neither a nor b is a Jack or if one is. There are then $\binom{2}{1} * \binom{12}{2} * \binom{4}{2} * \binom{4}{2} + \binom{2}{1} * \binom{2}{2} * \binom{12}{1} * \binom{4}{2} = 4752 + 144 = 4896$ hands of this form. With two wild cards, there are two possibilities. Both have form $WWabc$, but may have higher designation of either a Straight or a Flush. To count those from the Flush category, again we must consider the case where none of a, b or c is a Jack and the case where one is. Thus we have $\binom{2}{2} * \binom{12}{3} * 4 + \binom{2}{2} * \binom{2}{1} * \binom{12}{2}$ hands. Subtracting those that are Straight Flushes leaves $880 + 132 - 226 = 786$. Next the hands whose higher designation is a Straight have five cases: a is Ace to 6, a is 7 to 10 and b a Jack, a is 7 to 10 and there is no Jack, a is a Jack, and a is a Queen. Counting these gives $6 * \binom{2}{2} * \binom{4}{2} * 4^3 + 4 * \binom{2}{2} * \binom{2}{1} * \binom{3}{1} * 4^2 + 4 * \binom{2}{2} * \binom{3}{2} * 4^3 + 2 * \binom{2}{2} * \binom{3}{2} * 4^2 + \binom{2}{2} * 4^3$. Less the 226 Straight Flush hands gives $2304 + 384 + 768 + 96 + 64 - 226 = 3390$ hands. Thus there are a total of 55,536 Three of a Kind hands.

Two Pair: For $aabbc$ hands, the two cases are when a is a Jack and when neither a nor b is a Jack. This totals $1 * \binom{12}{1} * \binom{4}{2} * 44 + \binom{12}{2} * \binom{4}{2} * \binom{4}{2} * 42 = 102,960$ natural Two Pair hands. With one wild card, the hand $Waabc$ counts as Two Pair. There are three cases here: a is a Jack, a is not a Jack but b is, and none are Jacks. This gives $\binom{2}{1} * 1 * \binom{12}{2} * 4^2 + \binom{2}{1} * \binom{12}{1} * \binom{4}{2} [\binom{11}{1} * 2 * 4 + \binom{11}{2} * 4^2] = 141,504$ hands. With two wild cards, the hand $WWabc$ when the higher designation is Three of a Kind is counted here. There are 16,192 hands of this type ($\binom{2}{2} * \binom{12}{3} * 4^3 + \binom{2}{2} * 2 * \binom{12}{2} * 4^2$). After subtracting those that have higher designation Straight Flush, Flush and Straight, we have $16,192 - 226 - 786 - 3390 = 11790$ hands. The total number of Two Pair hands is then 256,254.

One Pair: With no wild cards, we have $aabcd$ where a is a Jack, where a is not a Jack but b is, and where a is not a Jack and neither are b, c, d. This gives $1 * \binom{12}{3} * 4^3 + \binom{12}{1} * \binom{4}{2} * [\binom{2}{1} * \binom{11}{2} * 4^2 + \binom{11}{3} * 4^3] = 901,120$ hands. With one wild card, the hands of the form $Wabcd$ with higher designation Flush or Straight are counted as One Pair hands. For those designated Flush, the cases are a is a Jack or none are Jacks. This yields $\binom{2}{1} * \binom{2}{1} * \binom{12}{3} + \binom{2}{1} * \binom{12}{4} * 4 - 276 = 4564$ hands. Considering the Straight hands leads us to four cases: a is Ace to 6, a is 7 to 10 with b a Jack, a is 7 to 10 with no Jacks, and a is a Jack. Remembering to subtract the Straight Flush hands gives $6*\binom{2}{1}*\binom{4}{3}*4^4+4*\binom{2}{1}*\binom{2}{1}*\binom{3}{2}*4^3+4*\binom{2}{1}*1*4^4+2*\binom{2}{1}*1*4^3-276 = 17,388$. This totals 923,072 One Pair hands.

Counting with Two Jacks Wild using the Wild Card Rule and the Revised Hand Ranking Nothing changes.